Appendix A

Formal Probability Calculus
For Metaphysics by Default

Supplement to the Public Essay at
http://mbdefault.org

version 1.1

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Reference to the Public Essay

This formal derivation does not duplicate the lengthy arguments of the public essay. Instead, it focuses on the mathematics of Metaphysics by Default, treating the essay's axioms as a formal and abstract system. For justification of the axioms, the reader should refer to the public essay itself, at mbdefault.org; with special attention to Chapters 9-16.

Axioms

Axiom 1. Randomness

The timings of birth and death will be assumed random. Neither the time of any birth nor the time of any death will be known in advance. The timings will form a simple, random distribution.

Axiom 2. Steady-State

The population exists in a steady-state environment, where conditions of existence are invariant over time. Hence the population will be seen to remain near a constant average value, when viewed over a long period of time.

Axiom 3. Continuity

Each person is born at a single, definite time; and each person passes away at a single, definite time. Between those two times, the person exists continuously; with no metaphysically significant change in composition or personal identity during that interval of existent time.

Here a simple timeline is introduced to illustrate the three axioms stated above.

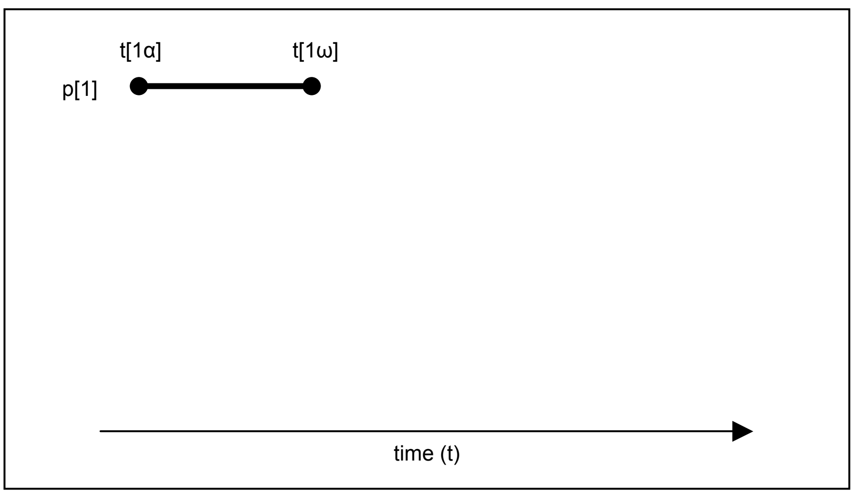


Figure 1

In Figure 1 we see a single person, p[1]. The rectangular bounding box encloses all space in the person's hypothetical environment. No persons exist unless drawn explicitly inside the bounding box of that environment.

Time (t) flows from left to right. Only one person, p[1], exists. That person is shown above the timeline.

p[1] is created at time t[1α].

p[1]'s existence continues as time flows to the right.

p[1] passes away at time t[1ω].

Axiom 4. Existential Passage

> When a person passes away, that person's existential "moment" is suspended in mortal amnesia until such time as a new person is born. At the time of birth, the suspended existential moment is "granted," subjectively, to the newborn "recipient," according to this condition: the receiving newborn must be born ***after*** the death of the granting person. This subjective event is known as "existential passage."

Figure 2 illustrates the simplest existential passage.

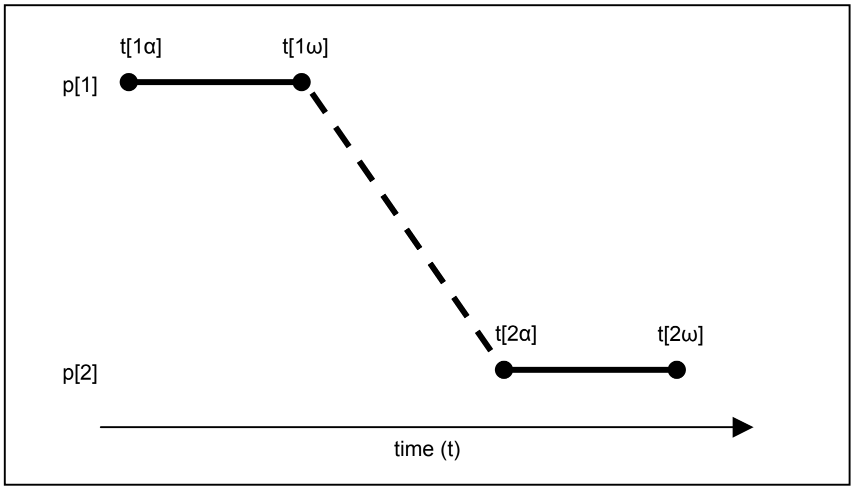


Figure 2

In Figure 2 person p[1] is shown again. Additionally, another person, p[2], has been added.

We can see that at time t[1ω] person p[1] passes away. According to Axiom 4 we assume p[1]'s existential moment is "suspended" in mortal amnesia at t[1ω]. Also, we can see that p[2] is born after the death of p[1]. This implies that t[2α] occurs ***after*** t[1ω].

Therefore, we can say that in Figure 2 p[1] "grants" existential passage to p[2]. p[2] is therefore the "recipient" of the existential passage. This all follows from Axiom 4.

This existential passage is illustrated in Figure 2 as a dashed line running diagonally from grantor to recipient. (As per Chapter 9 of the public essay at mbdefault.org, the dashed line merely symbolizes the subjective event. No "thing" transfers between p[1] and p[2].)

Axiom 5. Unique Recipient of an Existential Passage

> Axiom 5 is a restriction on Axiom 4: The only person who can receive an existential passage is the ***first*** person created ***after*** the grantor's death.

Axiom 6. The Null Condition

> **Axiom 6A:**
> If at any time no recipient person exists who can satisfy the axioms for receipt of an existential passage (Axioms 4 and 5), then ***no*** existential passage occurs at that time. This is a Null Condition.

Axiom 6B (*ex nihilo* passage):
Likewise, if a recipient person is born at a time when no deceased person meets the condition of Axiom 4, then a Null Condition also occurs. The putative recipient in this case receives no existential passage at creation. This is known as *ex nihilo* passage.

The next three figures illustrate Axioms 5 and 6:

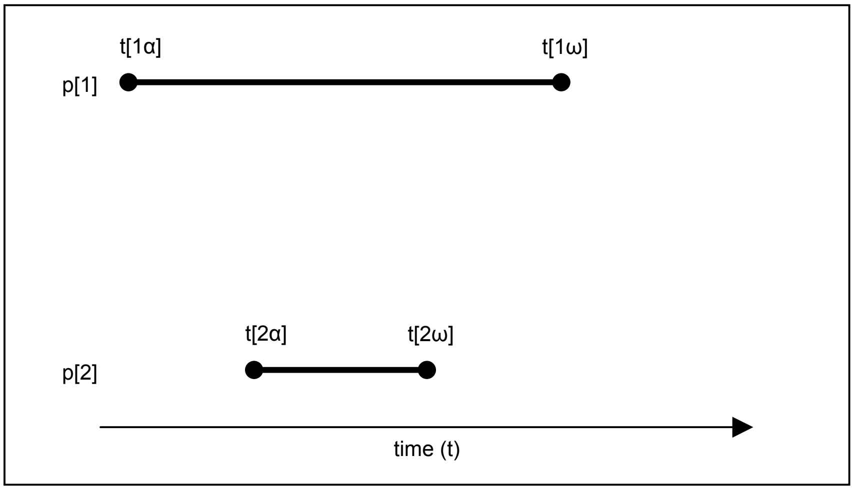


Figure 3

In Figure 3 we see that p[1] passes away at time t[1ω]. Also, we see that no person is born after t[1ω]. This results in a Null Condition according to Axiom 6A.

Likewise, no person is born after p[2]'s death at time t[2ω]. Axiom 6A denies the existential passage, either from p[1] to p[2], or from p[2] to p[1].

So no existential passage is granted in Figure 3.

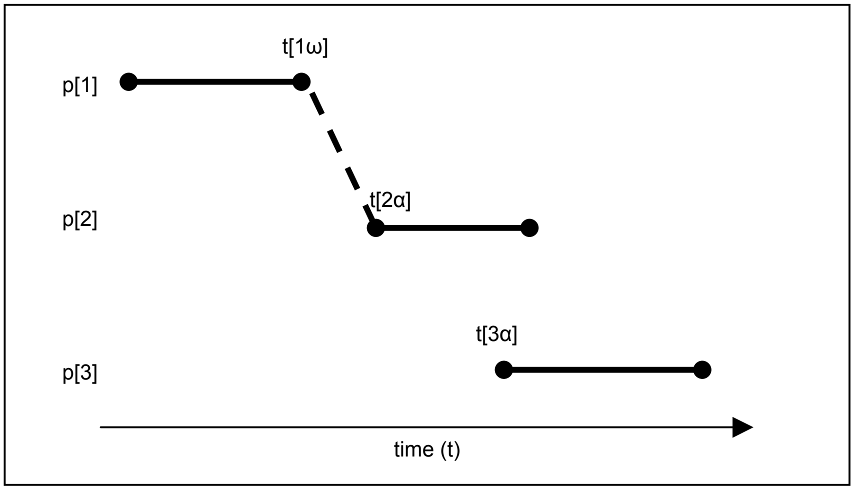


Figure 4

In Figure 4, p[1] passes away at time t[1ω]. p[2]'s time of birth, t[2α], falls after t[1ω] and before t[3α]. Hence, by Axioms 4 and 5 p[1] passes to p[2].

p[3] receives no existential passage, according to the Null Condition of Axiom 6B. The birth of p[3] is an *ex nihilo* passage.

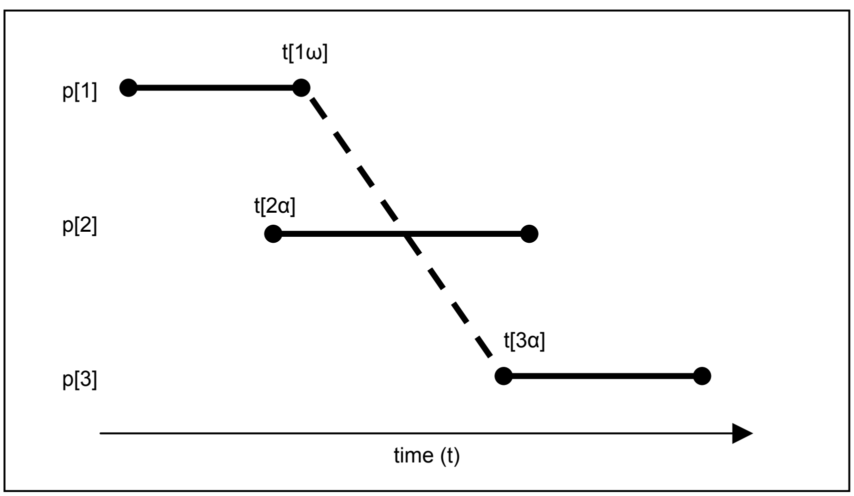


Figure 5

In Figure 5, p[1] passes to p[3] according to Axioms 4 and 5.

p[2] is bypassed, and is granted no existential passage, according to Axiom 6B. The birth of p[2] is therefore an *ex nihilo* passage.

Note that the timeline does not show spatial relations of the persons, but only temporal relations. Spatial relations will be considered ***irrelevant*** to the persons. Existential passages will be assumed to work irrespective of any distance between persons. We'll formalize this assumption with another axiom:

Axiom 7. Action at a Distance

> All passage relations are strictly temporal, and are irrespective of distances between persons. They operate over any distance, instantaneously, with no preference for each person's location.

Again, reviewing Figure 5: In this Figure p[3] may be either spatially near to p[1], or far from p[1]. Regardless, p[3] ***must*** receive the passage from p[1], according to Axioms 4 and 5. This is only re-affirmed by Axiom 7.

Axiom 8. Merged Passages

> Figure 2 illustrates a "unitary passage," wherein one person passes to another. Axiom 5 can sometimes force situations wherein ***several*** persons must pass to the ***same*** recipient. These events will be called "merged passages."

Figures 6, 7, 8 and 9 illustrate some merged passages.

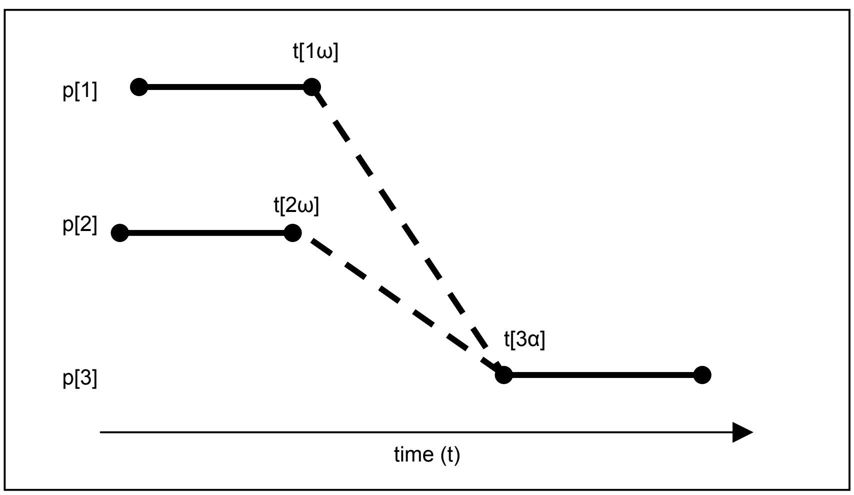


Figure 6

Figure 6 illustrates a "2-to-1 passage." Both p[1] and p[2] pass to p[3].

p[3] is born at time t[3α]. p[3] is the first person born after the death of p[1] at t[1ω], and also after the death of p[2] at t[2ω].

So both p[1] and p[2] must pass to p[3] according to Axioms 5 and 8.

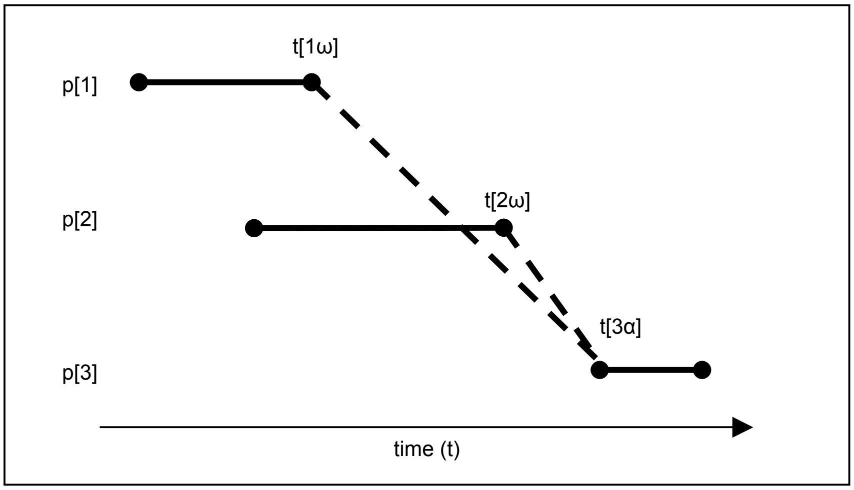


Figure 7

Figure 7 illustrates another 2-to-1 merged passage. Again, according to Axioms 5 and 8, p[3] must receive the passages of p[1] and p[2].

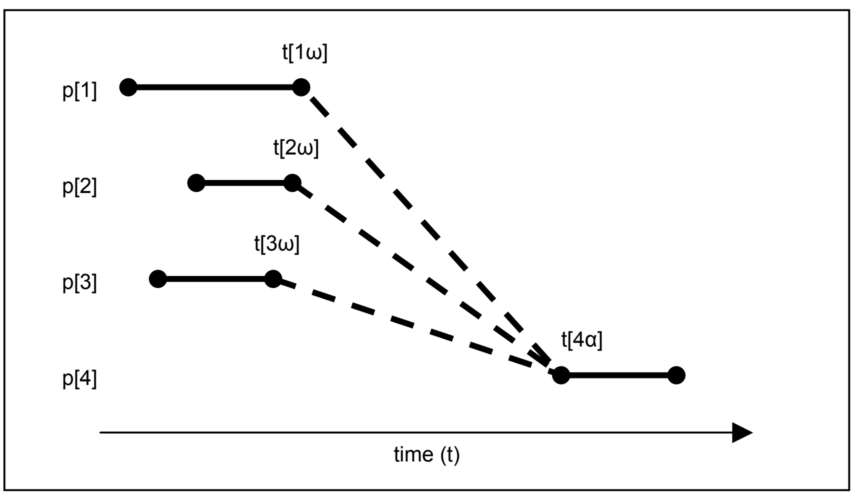


Figure 8

Figure 8 illustrates a "3-to-1 merged passage." p[4] satisfies Axioms 5 and 8 for p[1], p[2] and p[3]. Hence, all three must pass to p[4] in a 3-to-1 merged passage.

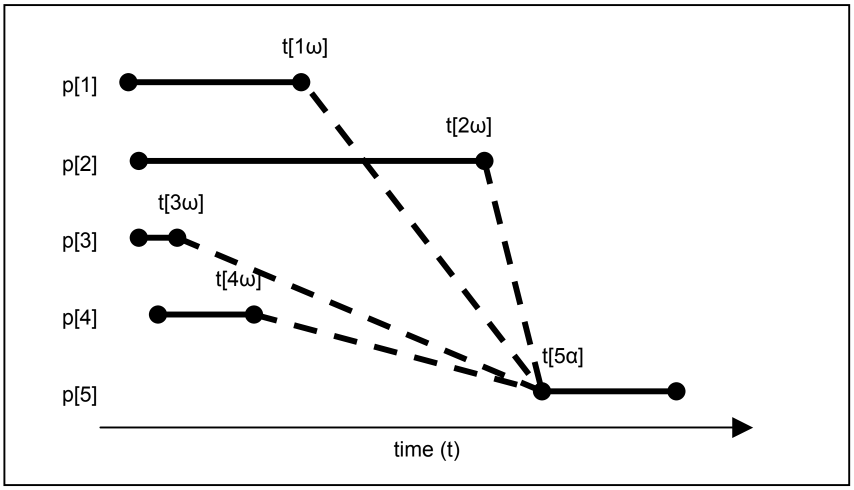


Figure 9

And Figure 9 illustrates a "4-to-1 merged passage." p[1], p[2], p[3], and p[4] must pass to p[5], according to Axioms 5 and 8.

Axiom 9. No Split Passage

> It will be assumed that no two events can occur at exactly the same time. That is to say, there will be no "synchronous" events. Hence, no passages will be split among multiple recipients.

Figure 10 illustrates Axiom 9, by explicitly superimposing the circular international *"not"* symbol over a disallowed split passage. (The symbol is reversed to improve the figure's legibility.)

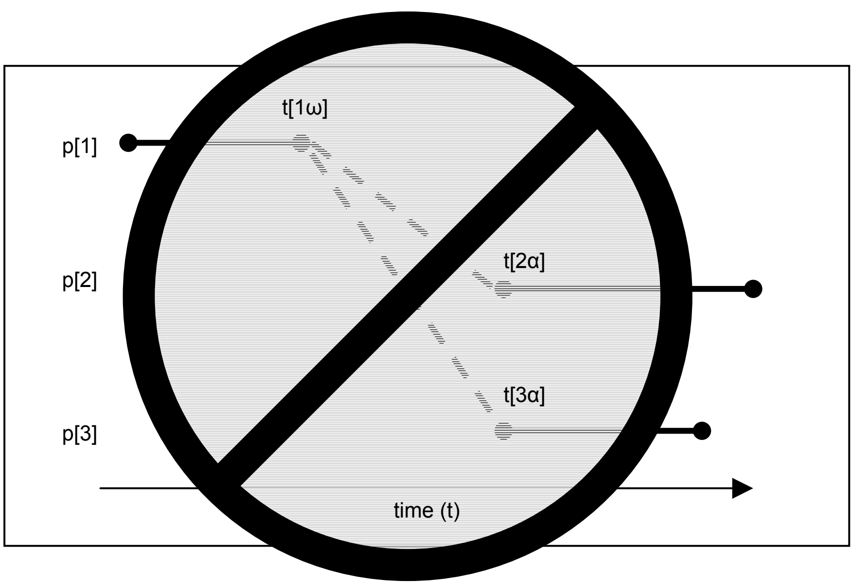


Figure 10

If we were to suppose the contrary to Axiom 9, and allow synchronous events; then in Figure 10 p[2] and p[3] would have synchronous births, t[2α] and t[3α] being the exact same time. In this case p[1] would be forced to split passage between p[2] and p[3], according to Axiom 5.

Axiom 9 assumes synchronous events to be impossible. Therefore we assume p[1]'s passage must go to just one recipient; that is, whichever one person, p[2] or p[3], will be determined at random to have been born ***first*** after t[1ω]. In effect, Axiom 9 will force t[2α] and t[3α] to be different times. This is the meaning of the superimposed *"not"* symbol in Figure 10: Synchronous events will be assumed not to occur.

Here it is worth noting that Axiom 9 may not hold true in the real world. If nature should relax the conditions of synchronization through some unknown function, then split passages could be possible. Their occurrence in nature would seem exceedingly rare at best, but no barrier other than the practical difficulty of synchronization would prevent them. (See Chapter 11 of the pubic essay for a fuller treatment of this possibility.) All the same, if we are to obtain quantitative mathematical results from the current analysis, Axiom 9 is a formal requirement.

These nine axioms set up the problems, which follow:

The Problems

(P1) Determine the absolute probability of several passage types, starting with the sole "unparticipated" type:

0-to-1 (*ex nihilo*)

and continuing to the first five participated types:

1-to-1 (unitary),
2-to-1,
3-to-1,
4-to-1, and
5-to-1.

The solution of (P1) will generate the numeric values needed to complete the following table:

passage type	absolute probability
0-to-1 (*ex nihilo*)	
1-to-1 (unitary)	
2-to-1	
3-to-1	
4-to-1	
5-to-1	

(P2) Determine the relative probability of merged passage with respect to unitary passage.

The solution of (P2) will generate the numeric values needed to derive the following ratio:

ratio	relative probability
merged : unitary	

Developing the Algorithm

Axiom 2, the steady-state axiom, is ambivalent and in need of clarification before formal solutions to the problems can be found. One way to clarify Axiom 2 is to make the population finite: The use of the phrase "steady-state" in the literature of probability calculus is consistent with a finite number of states, where a population would be explicitly prohibited from exceeding a finite mark, although events may continue indefinitely. This finite definition also allows us to manually calculate some reasonable probabilities, which is not possible in the infinite case.

This also makes sense for equilibrium reasons. In a process that may continue forever, with possibly an infinite number of participants, the non-simultaneity assumption of Axiom 9 becomes significant, and the limiting population or equilibrium probabilities become more elusive. If the population could grow to infinity, one could not guarantee that the population would remain "steady" after an infinite number of steps.

For these reasons it is necessary to amend Axiom 2:

Axiom 2A. Steady State / Random Limit

For some finite number N, the environment is full, at which point it is guaranteed that a person in the environment will pass away before a new person is born.

Let $U_j \; \{ j=0, 1, \ldots n \}$ denote that there are currently *j* persons in the environment. Although the events marking birth and death of persons are time dependent, we are only interested, for the sake of calculating probabilities, in the sequence of increments and decrements and not the temporal qualities. For this reason, we only calculate probabilities of *changed* states: not those for unchanged states. Thus, although the environment may remain in a certain state U_i from one time unit to the next, we will only consider events that result in a change from U_i :

either $U_i \longrightarrow U_{i+1}$ or $U_i \longrightarrow U_{i-1}$.

Let's consider the changes in state which are possible when we begin at state U_i . We will go either to U_{i-1} or U_{i+1} , with probabilities q_i or p_i respectively.

$$U_{i-1} \xleftarrow{q_i} U_i \xrightarrow[p_i]{} U_{i+1}$$

Also, these are the only possible events; and so their combined probabilities always sum to 1.

$$p_i + q_i = 1$$

There must be a distinct *p* and *q* for all states of the system. And so we have this graph for the system:

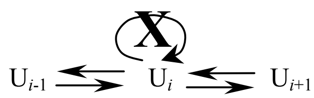
(1) $$U_0 \underset{p_0}{\overset{q_1}{\rightleftarrows}} U_1 \underset{p_1}{\overset{q_2}{\rightleftarrows}} U_2 \underset{p_2}{\overset{q_3}{\rightleftarrows}} \cdots \underset{p_{N-2}}{\overset{q_{N-1}}{\rightleftarrows}} U_{N-1} \underset{p_{N-1}}{\overset{q_N}{\rightleftarrows}} U_N$$

$q_0 = 0$ and $p_0 = 1$ } 0 is the minimum number of persons.
$q_N = 1$ and $p_N = 0$ } N is the maximum number of persons.

The system is reflected at U_0 and U_N . That is to say:

$U_0 \rightarrow U_1$ and $U_N \rightarrow U_{N-1}$, both with a probability of 1.

Also, because the system only considers changes to the states U, no state transitions to itself. Graphically, we deny this by placing an "**X**" on the disallowed self-transition:



This condition will correspond later on as "zeros on the diagonal" of the transition matrix for this system.

Calculating Probabilities

The system's transition matrix will be essential to the probability calculations. To set up a transition matrix, it will be necessary to choose an arbitrary "equilibrium point" so as to fix the matrix values. An example shows how these values are derived:

Let's suppose that an environment is guaranteed to hold only a finite number of persons, as according to Axiom 2A. And after recording the results of a sufficiently high number of events we notice that when there are ***four*** persons the environment has reached an equilibrium point, wherein exactly 50% of the time, one or the other of two conditions occurs:

(B) *Birth:* A new person is born somewhere in the environment *before* one of the environment's existing persons passes away.

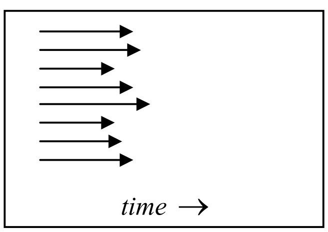
(D) *Death:* One of the four persons passes away before a new person is born *into the four-person environment*.

Thus at the equilibrium point the probability of (B) equals the probability of (D):

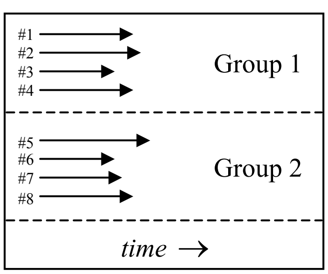
$P(\mathrm{B}) = P(\mathrm{D}) = 1/2$ at the chosen equilibrium point (U_4).

From this equilibrium starting point, we can deduce the probabilities which will apply when more persons are added to the same environment. For example, we can consider what would happen if the population were to double, moving the environment from four persons to eight (U_8). When there are *eight* persons, $P(\mathrm{B}) = 1/3$, by the following reasoning:

Consider the timeline illustration below:



Eight persons are here in the environment. If we divide in half, as below:



we get two sub-environments of four persons each. We call them Group 1 and Group 2. If we consider the persons in each group distinctly, we get two distinct versions of (D):

(D_1) One of the four persons (#1, #2, #3 or #4) passes away before a new person is born anywhere in the environment of Group 1.

(D_2) One of the four persons (#5, #6, #7 or #8) passes away before a new person is born anywhere in the environment of Group 2.

We know from the definition of the equilibrium point that $P(D_1) = 1/2$.

Likewise we know that $P(D_2) = 1/2$.

A condition (D) can occur independently either in Group 1 or Group 2. Neither (D_1) nor (D_2) determines the overall state of the entire environment – only the state of that group of four persons in which (D) has occurred. (If a person passes away in Group 1 before the next birth, it does not necessarily follow that a person will pass away in Group 2 before the next birth.)

But if (B) occurs in either Group 1 or Group 2, (B) occurs for the entire environment. *Any birth* determines the state of the entire environment. By explicit definition, that birth need occur only once, anywhere in the environment, to force the entire environment to (B).

And so by this reasoning:

$P(B) = P(D_1) = P(D_2)$

while the sum of all probabilities must sum to 1. Therefore,

$P(B) + P(D_1) + P(D_2) = 1$

So while the environment is in a state of eight persons (U_8),

$P(B) = 1/3$ and $P(D) = 2/3$ { where D is the event D_1 or D_2 }

Now, condition (D) can be restated in terms of existential passages. In a four-person environment, condition (D) is precisely a 4-to-1 merged passage. In an eight-person environment, condition (D) is precisely an 8-to-1 merged passage. And in an environment with N persons, condition (D) equates with an *n*-to-1 merged passage. So these conditions will map to a transition matrix of the system's passage probabilities.

We will use these equilibrium results to prepare a transition matrix which will churn out the particular numeric values needed to solve problems (P1) and (P2). But before we can use the transition matrix, we must first derive a theorem which will render the matrix values meaningful.

Theorem for Calculating Probabilities

(with a proof to follow)

The absolute probability of an *n*-tuple passage in an N-person system is:

(2) (for $n = 0$)

$$P(\text{0-passage}) = \frac{\sum_{i=1}^{N-1} P(U_i^0)\, p_i}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

(3) (for $N \geq n > 0$)

$$P(n\text{-passage}) = \frac{\sum_{i=0}^{N-n} P(U_{i+n}^0)\, p_i\, q_{i+1}\, q_{i+2} \ldots q_{i+n}}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

where:

P(0-passage) = the probability of an *ex nihilo* passage, or a "grant of zero passages."

P(n-passage) = the probability of an n -tuple passage, or a "grant of n passages."

p_i = probability of event B_j (going from state U_j to U_{j+1} -- a birth).

q_i = probability of event D_j (going from state U_j to U_{j-1} -- a death).

$P(U_j^k)$ = probability of being in state U_j (j persons) when the most recent birth is k events removed. Said another way, the last k events were deaths.

We have introduced some notation for the types of passages. Three different notational terms will be used throughout the equations, and their verbose equivalents should be remembered:

P(0-passage) := the probability of an *ex nihilo* passage.
P(n-passage) := the probability of an n-tuple passage.
P(any passage) := the probability of any passage.

Now, stating the meaning of (2) and (3) (again, with a proof to follow):

(2) states the absolute probability of an *ex nihilo* passage as the sum of the probabilities of independent *ex nihilo* passages, divided by the sum of the probabilities of any passage.

(3) states the absolute probability of an n-tuple passage as the sum of the probabilities of independent n-tuple passages, divided by the sum of the probabilities of any passage.

Note that (2) is just a special case of (3).

Eventually we will choose numeric values for the terms and calculate the probabilities. But first it is necessary to present a proof of the theorem. First we prove (2), which calculates the *ex nihilo* passage probability. Then we prove the more general (3), which calculates the higher-order passage probabilities.

The proof will require the following result: No matter what the starting condition, after a sufficiently large number of steps the probability of being in state U_j remains constant at λ_j. In other words:

(4) $P(U_j) = \lambda_j$

This result follows from the definition of the limiting transition matrix Λ, where:

$$\Lambda = \mathsf{E}\text{-}\lim_{n\to\infty} Q^n = \left[\lambda \mid \lambda \mid \lambda \mid \cdots \mid \lambda\right]$$

This is the definition of the Euler Limit. This definition, labeled as number (5), is provided in the following section, "Reference Definitions and Theorems."

Reference Definitions and Theorems

The definitions and theorems of this calculus follow from John G. Kemeny and J. Laurie Snell, *Finite Markov Chains* (Princeton: D. Van Nostrand Company, 1960) 25, 35-39, 99-100.

For additional references to these topics see William Feller, *An Introduction to Probability Theory and Its Applications*, vol. I, 2nd edition (New York: John Wiley & Sons, 1957). See especially "Waiting Line And Servicing Problems," in Feller 413-21.

***Definition: n*-th Step Transition Matrix Probabilities**

(from Kemeny 25, Definition 2.1.2)

The n-th step transition matrix probabilities for a Markov process, denoted by $p_{ij}(n)$ are:

$p_{ij}(n) = P[f_n = U_j \mid f_{n-1} = U_i]$, where:

$p_{ij}(n)$:= the probability of going from i to j

$P[f_n = U_j \mid f_{n-1} = U_i]$:= the probability of being in U_j at the nth step, given that the system was in U_i at the n-1 step

We have simplified Kemeny's notation. In simplified notation:

$p_{i,i+1} = p_i$
$p_{i,i-1} = q_i$
$p_{i,j} = 0$, for $j \neq i \pm 1$

***Definition:* Finite Markov Chain**

(from Kemeny 25, Definition 2.1.3)

A finite Markov Chain is a finite Markov process such that the transition probabilities $p_{ij}(n)$ do not depend on n. (That is to say, the process is time-independent, so we can drop n.) In this case they are denoted by p_{ij} . The elements (U) are called states.

***Definition:* Transition Matrix for a Markov Chain**

(from Kemeny 25, Definition 2.1.4)

The transition matrix for a Markov Chain is the matrix P with entries p_{ij} . The initial probability vector is the vector

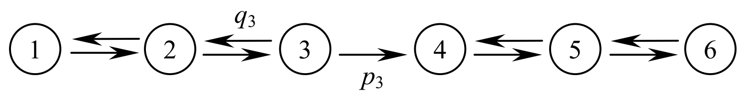
$$\pi_0 = \{ P_j^0 \} = \{ P[f_0 = U_j] \}$$

where U_j is the initial state.

***Definition:* Ergodic Set**

(from Kemeny 35-39, Section 2.4)

An ergodic set is the set of states that can travel within their set. They "communicate." *e.g.:*



Here { 1, 2, 3 } and { 4, 5, 6 } comprise two ergodic sets. A chain consisting of a single ergodic set is called an "ergodic chain." The system under consideration in the current problem is an example of an ergodic chain.

***Theorem:* Euler-Summable Sequence**

(from Kemeny 99-100, Theorem 5.1.1)

Given a sequence { S_i }, let

$$w_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} (1-k)^i S_i \qquad \text{for some } 0 < k < 1$$

If the sequence $w_1, w_2, \ldots, w_n, \ldots$ converges to w, then the original sequence { S_i } is Euler-summable.

This theorem is developed further:

***Theorem:* Euler-Summable Limiting Matrix**

(from Kemeny 99-100, Theorem 5.1.1)

For any ergodic chain the sequence of powers P^n is Euler-summable to a limiting matrix A, and this limiting matrix is of the form A = $\xi\alpha$, where α is a position vector.

$$A = \left[\alpha \;\middle|\; \alpha \;\middle|\; \alpha \;\middle|\; \cdots \;\middle|\; \alpha\right]$$

For example, if:

$$\alpha = \begin{bmatrix} .25 \\ .4 \\ .15 \\ .1 \\ .1 \end{bmatrix} \qquad \text{then} \qquad A = \begin{bmatrix} .25 & .25 & .25 & .25 & .25 \\ .4 & .4 & .4 & .4 & .4 \\ .15 & .15 & .15 & .15 & .15 \\ .1 & .1 & .1 & .1 & .1 \\ .1 & .1 & .1 & .1 & .1 \end{bmatrix}$$

***Theorem:* Replacing Limiting Matrix With Euler-Limit**

(from Kemeny 100, Theorem 5.1.2)

If P is an ergodic transition matrix, and A and α as above, then:

(a) For any probability vector π, the sequence P_π^n is Euler-summable to α .

(b) The vector α is the unique fixed probability vector of P.

(c) PA = AP = A (Which is the same as saying $Q\Lambda = \Lambda$.)

These two theorems can be interpreted as saying that the long-range predictions are independent of the initial vector π_0. By solving Q $\lambda = \lambda$ we have found the unique solution, as in (b); that gives the probability after a large number of trials, irrespective of the initial vector π_0, *i.e.:*

$$\lambda = \lim_{n\to\infty} \binom{n}{i} k^{n-i} (1-k)^i Q^n \pi_0$$

Our Euler-summable limiting matrix becomes the Euler Limit (E-lim) of Q^n, rather than the limit itself:

(5) $$\Lambda = \mathsf{E}\text{-}\lim_{n\to\infty} Q^n = \left[\lambda \mid \lambda \mid \lambda \mid \cdots \mid \lambda\right]$$

***Note:* Application of Euler Limits to Kemeny's Finite Markov Chain**

Suppose, for simplicity, we have N=2 (at most two persons in the system), and a resulting 3×3 matrix Q.

$$Q = \begin{bmatrix} 0 & p & 0 \\ 1 & 0 & 1 \\ 0 & q & 0 \end{bmatrix}$$

with $\alpha = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that Q $\alpha = \alpha$. (Note that all columns add to 1.)

Consider first an initial position vector $\pi_0 = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

Which is to say that the probability is p that there will be 0 persons at the start; q that there will be 1 persons at the start; and r that there will be 2 persons at the start.

The vector is such that $q \neq 0$ and $q \neq 1$, so it is not the case that both $p = 0$ and $r = 0$.

Apply Q to π_0. After one event, the system will be in state $Q\pi_0$. After two events, it will be in $Q^2\pi_0$. After three events, it will be in $Q^3\pi_0$. And so on. The limit of $Q^n \pi_0$ as $n \to\infty$ will actually be the vector α ; *i.e.,*

$$\lim_{n\to\infty} Q^n \pi_0 = \alpha$$

So in this simplified case we don't need the Euler Limit; the actual limit can be obtained. To obtain that limit, we consider π_1 with one of the following possibilities:

(a) $p = 1$, or
(b) $q = 1$, or
(c) $r = 1$

We will choose (a), arbitrarily. So: $\pi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We start with no persons, so the next event must add one person: $Q\pi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The population can go either up or down: $Q^2\pi_1 = \begin{bmatrix} p \\ 0 \\ q \end{bmatrix}$

Thereafter the population must return to one: $Q^3\pi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

This pattern repeats indefinitely:

$$(6) \qquad Q^{2n}\pi_1 = \begin{bmatrix} p \\ 0 \\ q \end{bmatrix}, \quad Q^{2n+1}\pi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \forall\, n$$

Again, we state the Euler Limit for this system (which is the same as the actual limit):

$$\mathsf{E}\text{-}\lim_{n\to\infty} Q^n \pi_0 = \alpha$$

and we see from (5) that the limit is an "average" of the two states:

$$\begin{bmatrix} p \\ 0 \\ q \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Our solution, Q, is in fact a two-cycle ergodic transition matrix (for any N). This only becomes apparent when we start in a state π_1 with either

(i) At the start there is assuredly an *even* number of persons, or
(ii) At the start there is assuredly an *odd* number of persons.

e.g.: $\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$ will alternate; as will $\begin{bmatrix} p \\ 0 \\ q \end{bmatrix}$, provided $p + q = 1$.

In solving for $Q\lambda = \lambda$, we are simply substituting the vector α with the equivalent vector λ. And so we see that at the limit, the solution may actually alternate between two vectors that average to λ. To use the desired result, we may have to assume that: { Not (i) and Not (ii).} That is, we will assume that there is at least a small probability ε_1 that the system started with an odd number of persons; and at least a small probability ε_2 that the system started with an even number of persons. Granted this assumption, we can solve the system.

Proof of Theorems

Returning now to the theorems to be proved (and problems to be solved). Restating (2), which is the theorem giving the absolute probability for *ex nihilo* passages:

$$(2) \quad (\text{for } n = 0) \qquad P(\text{0-passage}) = \frac{\sum_{i=1}^{N-1} P(U_i^0)\, p_i}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

To prove (2) we will derive the denominator of the equation first, and then the numerator.

Deriving the denominator:

Denominator = the sum of the probabilities of any passage

We can calculate the probability that the next event will be a birth; resulting in any passage. From our reference definitions and theorems we know that our limiting state is stable, *i.e.* $Q\Lambda = \Lambda$. And so the probability that the next event will be an *ex nihilo* passage is exactly the same as the probability that the last event was any passage.

Looking at it another way: If the last event was a passage, we know that the last event corresponded with a birth, as only a birth event can cause a passage. That birth effectively removed all passage participants from their states of mortal amnesia, wiping the slate clean. No persons then remained in mortal amnesia to participate in any passages thereafter. So if that "slate-cleaning" birth is followed immediately thereafter by *another* birth, then the *second* birth must produce an *ex nihilo* passage; as no persons remain in mortal amnesia to participate in that second passage.

So deriving the denominator: The probability that the next event will be an *ex nihilo* passage is the same as the probability that "the slate has been cleaned" by some passage previously; such that "the queue has been emptied."

We get the sum total probability for any passage by summing all of the N possible states U_i for which any passage can occur:

$$(7) \qquad P(\text{any passage}) = \sum_{i=1}^{N} P(U_i^0)$$

The next event will be a birth, producing *ex nihilo* passage from state U_i to U_{i+1}. Expressed in the same notation as was used in (6), its probability is:

$$(8) \qquad P(\text{any passage}) = \sum_{i=0}^{N-1} P(U_{i+1}^0)$$

The probability of (8) is, as we've seen, exactly the same as that of (7). This proves the derivation of the denominator of (2).

Deriving the numerator:

Numerator = the sum of the probabilities of independent *ex nihilo* passages

The sufficient and necessary conditions for an *ex nihilo* passage at state U_i are that:

1. the event preceding U_i is a birth, and
2. the event at U_i is a birth.

$$U_{i-1} \xrightarrow{B_{i-1}} U_i \xrightarrow{B_i} U_{i+1}$$

Putting it another way: If B_{i-1} "emptied the queue" with a birth, then B_i, which is another birth, must produce an *ex nihilo* passage.

This requires that the state prior to B_{i-1} must have been U_{i-1} . Synonymously, we are in state U_i with a history of U_i^0. Again, U_i^0 states that U_i has "emptied the queue" of pending passages. The probability of an *ex nihilo* passage at state U_i can be expressed as the probability of a "grant of zero passages," or of a "0-passage" at U_i .

And this probability can be equated with the probability that the system has reached U_i through an *ex nihilo* passage, multiplied by the probability of event B_i. Expressing this probability as the product of two factors, it becomes:

$$P(U_i^0) \times p_i$$

We will symbolize the probability of event x at state y as: $P(x \mid y)$. Hence:

(9) $$\begin{aligned} P(\text{0-passage} \mid U_i) &= P(U_i^0)\, p_i \\ &= P(B_{i-1} \mid U_{i-1})\ P(U_{i-1})\ P(B_i \mid U_i) \\ &= (p_{i-1})(\lambda_{i-1})(p_i) \qquad \{ \text{ after (4) } \} \end{aligned}$$

The derived equation becomes:

(10) $$P(\text{0-passage} \mid U_i) = p_{i-1}\, \lambda_{i-1}\, p_i$$

which can be calculated from the limiting matrix Q^n as $n \to \infty$, after (5).

Also, we know that a given passage can be of only one type. Consequently, these events are independent; which is to say that an *ex nihilo* passage U_j at a given time is independent of an *ex nihilo* passage at U_k , provided $j \neq k$.

Additionally, these events are exhaustive (i.e., inclusive); because a given *ex nihilo* passage will always fit one of these categories, and no other.

So to finish deriving the denominator:

As shown previously:

$$P(\text{0-passage} \mid U_i) = P(U_i^0)\, p_i$$

An *ex nihilo* passage can only occur in states U_1 to U_{N-1} . And so the sum of the probabilities of independent *ex nihilo* passages is:

$$\sum_{i=1}^{N-1} P(U_i^0)\, p_i$$

This proves the derivation of the numerator of (2).

Conclusion of Proof of (2)

At this point both the numerator and denominator have been proved. Putting them together:

The absolute probability of an *ex nihilo* passage is the sum of the probabilities of independent *ex nihilo* passages, divided by the sum of the probabilities of any passage:

$$\text{(for } n = 0\text{)} \qquad P(\text{0-passage}) = \frac{\sum_{i=1}^{N-1} P(\,U_i^0\,)\, p_i}{\sum_{i=0}^{N-1} P(\,U_{i+1}^0\,)}$$

This proves (2).

Proof of (3):

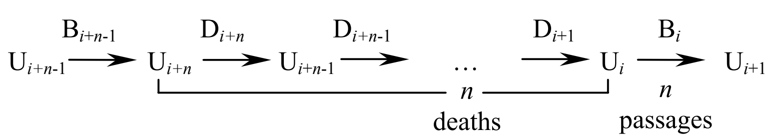
Restating (3), the theorem giving the absolute probabilities for unitary and merged passages:

$$\text{(3) (for } N \geq n > 0\text{)} \qquad P(n\text{-passage}) = \frac{\sum_{i=0}^{N-n} P(\,U_{i+n}^0\,)\, p_i\, q_{i+1}\, q_{i+2} \ldots q_{i+n}}{\sum_{i=0}^{N-1} P(\,U_{i+1}^0\,)}$$

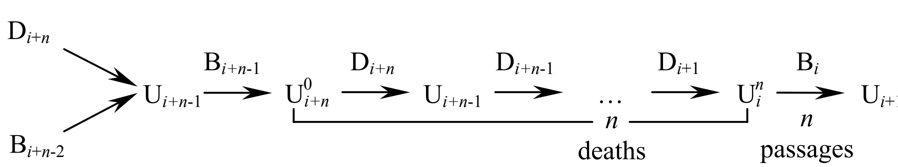
Sufficient and necessary conditions for an n-tuple passage at state U_i are that:

1. the last birth is n events removed, and
2. the event at U_i is B_i .

Graphically,



The history sequence requires the system be at state U_{i+n}^0 exactly n events prior to the current state, U_i^n . How the system arrives at U_{i+n-1} does not matter. Graphically,



Thus the probability of an n-tuple passage at state U_i can be stated as the product of four factors:

1. the probability of a birth at U_{i+n-1} (which equals that of an *ex nihilo* passage at U_{i+n}).
2. the probability of being in state U_{i+n-1} (or λ_{i+n-1}).
3. the probability of n deaths (each denoted as some D_j).
4. the probability of a birth at U_i .

This can be calculated from the limiting matrix Q^n as $n \to \infty$, again in terms of λ, p, and q, after (9) and (10).

(11) $P(n\text{-passage} \mid U_i) = P(B_{i+n-1} \mid U_{i+n-1}) \times P(U_{i+n-1}) \times P(D_{i+n})\, P(D_{i+n-1}) \ldots P(D_{i+1}) \times P(B_i \mid U_i)$

$$= P(U^0_{i+n}) \times q_{i+n}\, q_{i+n-1} \ldots q_{i+1} \times p_i \quad \text{\{ per the definition of factor 1, above. \}}$$

$$= \lambda_{i+n-1}\, p_{i+n-1}\, q_{i+n}\, q_{i+n-1} \ldots q_{i+1}\, p_i \quad \text{\{ after (9) and (10) \}}$$

Note that an n-tuple passage can only occur in states U_0 to U_{N-n}. And so we can state the absolute probability as:

$$P(n\text{-passage}) = \frac{\sum_{i=0}^{N-n} P(\, n\text{-passage} \mid U_i\,)}{P(\text{ any passage })}$$

$$= \frac{\sum_{i=0}^{N-n} P(\, U^0_i\,)\, q_{i+n}\, q_{i+n-1} \ldots q_{i+1}\, p_i}{\sum_{i=0}^{N-1} P(\, U^0_{i+1}\,)}$$

Since these events are independent and exhaustive, theorem (3) is proved.

Corollary: hexatuple+ passages

We can calculate the probabilities of *ex nihilo*, unitary, 2-to-1, 3-to-1, 4-to-1 and 5-to-1 passages directly, by theorem (3). Higher-order passages need not be calculated individually: Theorem (3) can be extended to calculate the sum total of hexatuple (6-to-1) and higher passage probabilities, as a whole.

The probability of a hexatuple or higher passage is equal to:

$$P(6^+\text{-passage}) = \frac{\sum_{i=0}^{N-6} P(\, U^0_{i+6}\,)\, q_{i+6}\, q_{i+5} \ldots q_{i+1}}{\sum_{i=0}^{N-1} P(\, U^0_{i+1}\,)}$$

$$= 1 - \sum_{i=0}^{5} d_i$$

where d_i is the absolute probability of an i passage, $i = 0, \ldots , 5$.

Proof:

The proof of the corollary is just the derivation of its ratio formula. This proof is straightforward.

For this equation, note that in the proof of Theorem (3) we calculated $P(n\text{-passage} \mid U_i)$ and summed over i to get the absolute probability. Here we first need to determine the relative probability equation for higher-order passage probabilities. This is the numerator of the corollary:

relative probability – numerator:

$$
\begin{aligned}
P(6^{+}\text{-passage}) &= P(6\text{-passage} \mid U_i) \\
&\quad + P(7\text{-passage} \mid U_{i-1}) \\
&\quad + P(8\text{-passage} \mid U_{i-2}) \\
&\quad + \ldots + P(6{+}i\text{-passage} \mid U_0) \\
&= P(U^0_{i+6})\, q_{i+6}\, q_{i+5} \ldots q_{i+1} [\, p_i \,] \\
&\quad + P(U^0_{i+6})\, q_{i+6}\, q_{i+5} \ldots q_{i+1} [\, q_i\, p_{i-1} \,] \\
&\quad + P(U^0_{i+6})\, q_{i+6}\, q_{i+5} \ldots q_{i+1} [\, q_i\, q_{i-1}\, p_{i-2} \,] \\
&\quad + \ldots + P(U^0_{i+6})\, q_{i+6}\, q_{i+5} \ldots q_{i+1} [\, q_i\, q_{i-1} \ldots q_1\, p_0 \,] \qquad \{ \text{ after } (11) \} \\
&= P(U^0_{i+6})\, q_{i+6} \ldots q_{i+1} [p_i + q_i\, p_{i-1} + q_i\, q_{i-1}\, p_{i-2} + \ldots + q_i\, q_{i-1} \ldots q_1\, p_0]
\end{aligned}
$$

At this point, the terms in brackets at right can be eliminated. This is because the bracketed terms are just the probabilities of the transitions which are possible from a starting point U_i.

p_i corresponds with an *ex nihilo* passage from U_i.
$q_i\, p_{i-1}$ corresponds with a unitary passage from U_i.
$q_i\, q_{i-1}\, p_{i-2}$ corresponds with a 2-to-1 passage from U_i.
And so on, up to:
$q_i\, q_{i-1} \ldots q_1\, p_0$, which corresponds with an *i*-to-1 passage from U_i.

As no other transitions are possible from state U_i , the sum of these probabilities must be 1. Continuing the derivation of the numerator's relative probability:

$$
\begin{aligned}
P(6^{+}\text{-passage}) &= P(U^0_{i+6})\, q_{i+6} \ldots q_{i+1} [\, 1 \,] \\
&= P(U^0_{i+6})\, q_{i+6} \ldots q_{i+1}
\end{aligned}
$$

This proves the numerator of the corollary.

The numerator is a relative probability. To get the absolute probability of 6^{+}-passage, we divide the numerator by the probability of *any* passage. Here, the denominator is the same as in Theorem (3), which has already been proved.

And so the corollary has been proved.

Numeric Result

At this point we would like to introduce the two parameters required to get an actual numeric answer, rather than a formula solution.

We take N, the upper limit of state, to be 12. This posits that the system will hold at most 12 persons.

$$N = 12$$

We take p_1 , the probability of B_1 , to be 4/5. This posits that the probability that a birth will transition the system from state U_1 to state U_2 is 4/5.

$$p_1 = 4/5$$

These two parameter values (N = 12 and p_1 = 4/5) have been chosen because they will allow us to get a first numeric answer with a relatively small amount of manual calculation; and hence provide a concise example of the technique. After we have worked through to a preliminary result using these two parameter values, we will change the parameters so as to produce a more *accurate* result. Only the summary results of these lengthier calculations will be presented.

So, on to preparations for a preliminary result:

At this point we can use the results of "Developing the Algorithm" and "Calculating Probabilities" sections. Axiom 2A , graph (1), and the equilibrium arguments come together to yield the following probability graph for this system:

q_i		1/5		1/3		3/7		1/2		5/9		3/5		7/11		2/3		9/13		5/7		11/15		1	
U_i	U_0	⇄	U_1	⇄	U_2	⇄	U_3	⇄	U_4	⇄	U_5	⇄	U_6	⇄	U_7	⇄	U_8	⇄	U_9	⇄	U_{10}	⇄	U_{11}	⇄	U_{12}
p_i		1		4/5		2/3		4/7		1/2		4/9		2/5		4/11		1/3		4/13		2/7		4/15	

Note that the equilibrium point has again been set arbitrarily at n=4. This choice has produced the same probability values around U_4 and U_8 as were derived before, in "Calculating Probabilities." We can see that the choice of p_1 = 4/5 will fit the progression of probability terms which have been determined by our choice of equilibrium point, and will therefore make for easy fractional calculations. (A different choice of equilibrium point would have called for a different choice of p_1, and vice versa.)

This graph can be expressed as a transition matrix $Q \in M_{13x13}$.

$$Q = \begin{bmatrix}
0 & 1/5 & & & & & & & & & & & \\
1 & 0 & 1/3 & & & & & & & & & & \\
 & 4/5 & 0 & 3/7 & & & & & \mathbf{0} & & & & \\
 & & 2/3 & 0 & 1/2 & & & & & & & & \\
 & & & 4/7 & 0 & 5/9 & & & & & & & \\
 & & & & 1/2 & 0 & 3/5 & & & & & & \\
 & & & & & 4/9 & 0 & 7/11 & & & & & \\
 & & & & & & 2/5 & 0 & 2/3 & & & & \\
 & & & & & & & 4/11 & 0 & 9/13 & & & \\
 & & & & & & & & 1/3 & 0 & 5/7 & & \\
 & & & \mathbf{0} & & & & & & 4/13 & 0 & 11/15 & \\
 & & & & & & & & & & 2/7 & 0 & 1 \\
 & & & & & & & & & & & 4/15 & 0
\end{bmatrix}$$

Let $\mathbf{v}$ be a vector in $\Re^{+13}$ (a positive 13-dimensional vector) with $|\,\mathbf{v}\,|_{sum} = 1$.

So $v_i \geq 0$ and $\Sigma\, v_i = 1$.

We can interpret **v** as a state vector, or probability vector. For example:

If $\mathbf{v}_0 = (\, 1, 0, 0, 0, 0, \ldots, 0\,)$ then we conclude the system is certain to be in state U_0 .

If $\mathbf{v}_2 = (\, 0, 0, 1, 0, 0, \ldots, 0\,)$ then we conclude the system is certain to be in state U_2 .

If $\mathbf{v}_x = (\, 0, 1/2, 0, 1/2, 0, \ldots, 0\,)$ then we conclude the system is equally likely to be either in state U_1 or state U_3.

Q is the transition matrix, the matrix of transition probabilities from the *current* state to the *next* state. To say that Q is the transition matrix, is to say that the product of matrix Q and the state vector **v** , (Q**v**), is the probable state of the system one event after starting in state **v** . e.g.,

$$Q\mathbf{v}_1 = \left[Q\right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = (\, \overset{1/5}{q_1}, 0, \overset{4/5}{p_1}, 0, \ldots, 0\,)$$

which denotes that we now have:

1. q_1 chance = 1/5, of being in state U_0 and
2. p_1 chance = 4/5, of being in state U_2 .

Note that $|\, Q\mathbf{v}\, |_{sum} = 1 = p_1 + q_1$.

And thus Q**v** can in turn be taken as a state vector. We can therefore calculate the probable state of the system at each subsequent step. For example, we can calculate the probable state of the system after two steps from state $\mathbf{v}_1$:

$$Q(Q\mathbf{v}_1) = Q^2\,\mathbf{v}_1 = (\,0,\ q_1 + p_1q_2,\ 0,\ p_1p_2,\ 0,\ \ldots,\ 0\,)$$

$$\qquad\qquad\qquad U_1 \qquad\quad U_3$$

Thus if we start in U_1 (position vector $\mathbf{v}_1$) then after two events (Q^2) we must either be in:

1. U_1 (with probability $q_1 + p_1q_2$), or
2. U_3 (with probability p_1p_2).

Notice again that the probabilities sum to 1:

$$\begin{aligned} |\,Q^2\mathbf{v}\,|_{sum} &= q_1 + p_1\,q_2 + p_1\,p_2 \\ &= q_1 + p_1(\,q_2 + p_2) \\ &= q_1 + p_1 \\ &= 1 \end{aligned}$$

We wish to solve the system Q $\boldsymbol{\lambda} = \boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is a position vector. We get the following equations:

$$\begin{aligned}
1/5\lambda_1 &= \lambda_0 \\
\lambda_0 + 1/3\lambda_2 &= \lambda_1 \\
4/5\lambda_1 + 3/7\lambda_3 &= \lambda_2 \\
2/3\lambda_2 + 1/2\lambda_4 &= \lambda_3 \\
4/7\lambda_3 + 5/9\lambda_5 &= \lambda_4 \\
1/2\lambda_4 + 3/5\lambda_6 &= \lambda_5 \\
4/9\lambda_5 + 7/11\lambda_7 &= \lambda_6 \\
2/5\lambda_6 + 2/3\lambda_8 &= \lambda_7 \\
4/11\lambda_7 + 9/13\lambda_9 &= \lambda_8 \\
1/3\lambda_8 + 5/7\lambda_{10} &= \lambda_9 \\
4/13\lambda_9 + 11/15\lambda_{11} &= \lambda_{10} \\
2/7\lambda_{10} + \lambda_{12} &= \lambda_{11} \\
4/15\lambda_{11} &= \lambda_{12}
\end{aligned}$$

The last equation ($4/15\ \lambda_{11} = \lambda_{12}$) is deducible from the others. Thus the equations above are N equations in N+1 unknowns. But using $\boldsymbol{\lambda}$ as a position vector, $|\,\boldsymbol{\lambda}\,|_{sum} = 1$, we get N+1 equations with N+1 unknowns, which can be solved with a unique solution, which is the Euler Limit.

$$\boldsymbol{\lambda} = \begin{bmatrix} 0.009 \\ 0.046 \\ 0.110 \\ 0.171 \\ 0.196 \\ 0.176 \\ 0.130 \\ 0.082 \\ 0.045 \\ 0.021 \\ 0.009 \\ 0.004 \\ 0.000 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \end{bmatrix}$$

Using the Algorithm

The algorithm incorporates Theorems (2) and (3), and the Corollary to Theorem (3) for 6^+-tuple passages.

(2) (for $n = 0$) $$P(\text{0-passage}) = \frac{\sum_{i=1}^{N-1} P(U_i^0)\, p_i}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

(3) (for $N \geq n > 0$) $$P(n\text{-passage}) = \frac{\sum_{i=0}^{N-n} P(U_{i+n}^0)\, p_i\, q_{i+1}\, q_{i+2} \ldots q_{i+n}}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

corollary to (3) $$P(6^+\text{-passage}) = \frac{\sum_{i=0}^{N-6} P(U_{i+6}^0)\, q_{i+6}\, q_{i+5} \ldots q_{i+1}}{\sum_{i=0}^{N-1} P(U_{i+1}^0)}$$

The algorithm invokes these theorems directly, calculating the probabilities of n-tuple passage types, from $n = 0$ to $n = 6+$. Thereafter, we just take a ratio of these absolute probabilities, in order to determine the probability of experiencing a merged passage, relative to a unitary passage.

An important trick of the algorithm lies in the order of steps applied. Normalization is not done until after computations are made. In fact, we multiply the vector by the l.c.m.("least common multiple"), ℓ, in order to "magnify" the vector. This allows us to compute with minimal rounding error. After we perform the computations, we divide by M to normalize the result. As in the flowchart illustration below:

$$\Lambda \longrightarrow \underset{\text{magnified}}{\ell\,\Lambda} \xrightarrow[\text{compute with } \ell\Lambda]{} \underset{\text{normalized}}{\frac{\ell\,\Lambda}{M}}$$

Other satisfactory algorithms may be possible.

Summary of Algorithm Steps

I. Chose a rational p_1 ($1/2 < p_1 < 1$)

II. Compute the terms of Q.

III. Solve $Q\Lambda = \Lambda$ in terms of λ_0, where λ_0 is the "free variable."

IV. Find ℓ = l.c.m. of { c_i }, and compute

$$\ell\,\Lambda = \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_N \end{bmatrix} \quad \text{and} \quad M = \sum n_i$$

V. Compute 9 × [N+1] table values; the "magnified values."

VI. Total the nine table columns ($t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8$). Divide these totals by M to normalize.

For each n-tuple passage multiply by the number of participants, n, and divide by the absolute probability of any passage, t_1.

For the special case of the 6^+-tuple passage, note that the total t_8 is computed by the corollary to Theorem (3).

VII. *Obtain the Solution of Problem (P1) -- Absolute Probabilities:* These nine normalized totals are precisely the following nine absolute event probabilities:

t_0/M = the absolute probability of *any* or *no* event (:= 1)
t_1/M = the absolute probability of experiencing any passage
t_2/M / (t_1/M) = the absolute probability of experiencing an *ex nihilo* passage
$1 \times t_3$ / (t_1/M) = the absolute probability of experiencing a unitary 1-to-1 passage
$2 \times t_4$ / (t_1/M) = the absolute probability of experiencing a 2-to-1 passage
$3 \times t_5$ / (t_1/M) = the absolute probability of experiencing a 3-to-1 passage
$4 \times t_6$ / (t_1/M) = the absolute probability of experiencing a 4-to-1 passage
$5 \times t_7$ / (t_1/M) = the absolute probability of experiencing a 5-to-1 passage
$6 \times t_8$ / (t_1/M) = the absolute probability of experiencing a 6^+-to-1 passage

VIII. *Obtain the Solution of Problem (P2) -- Relative Probability:* The probability of experiencing a merged passage, relative to that of experiencing a unitary passage, is just the sum of all absolute merger probabilities divided by the absolute unitary probability. Taking the formulae for these probabilities from step (VII), we get:

$(2t_4 + 3t_5 + 4t_6 + 5t_7 + 6t_8) / t_3$ = the relative probability of merged to unitary passage

Details of Algorithm Steps

The Roman numerals correspond with the numerals denoting each step in the previous section, "Summary of Algorithm Steps."

I. $p_0 = 1$, by definition, as the first event in an empty system must be a birth. $p_1 = 4/5$, chosen because it satisfies the requirement of (I), and because this fraction is relatively easy to work with. (We will use this fraction again when we complete the entire calculation, in the next section, "Parameters for a Preliminary Result.")

II. Compute the terms of Q.

$$Q = \begin{bmatrix} 0 & q_1 & & & & & \\ p_0 & 0 & q_2 & & & & \\ & p_1 & 0 & q_3 & & & \\ & & p_2 & \ddots & & & \\ & & & & & q_{k-1} & \\ & & & & p_{k-2} & 0 & q_k \\ & & & & & p_{k-1} & 0 \end{bmatrix}$$

Using $p_0 = 1$, $p_1 = 4/5$, we can obtain fractional values for the terms of Q.

To begin these calculations, we state p_1 as an expression in two variables, the numerator *a* and the denominator *b*. So:

$p_1 = 4/5 = a/b$ { where $a = 4$ and $b = 5$ }

Each birth event increases the population by one. Likewise, a comparison with the graph and transition matrix Q of the "Numeric Result" section shows that each birth event increases the denominator *b* by one as well. Expressing the fraction *a*/*b* more generally, we replace "*b*" with "*c*", so named because it will become a factor of the "***C***ommon denominator":

$p_n = a/(c+1)$ { where $p_{n-1} = a/c$ }

And so we find:

$p_2 = 4/6$
$p_3 = 4/7$
$p_4 = 4/8$
$p_5 = 4/9$ and so on.

And now moving to the terms q_n .

$q_i = 1 - p_i$ for i = 0, … , N

These formulae will make it possible for us to solve $Q\Lambda = \Lambda$.

III. Solve $Q\Lambda = \Lambda$ in terms of λ_0, where λ_0 is the "free variable."

Take λ_0 as the free variable for Λ in terms of λ_0.

QΛ = Λ

$$Q\Lambda = \begin{bmatrix} 0 & q_1 & & & & & \\ 1 & 0 & q_2 & & & & \\ & p_1 & 0 & q_3 & & & \\ & & p_2 & \ddots & & & \\ & & & & \ddots & q_{k-1} & \\ & & & & p_{k-2} & 0 & 1 \\ & & & & & p_{k-1} & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} q_1\lambda_1 \\ \lambda_0 + q_2\lambda_2 \\ p_1\lambda_1 + q_3\lambda_3 \\ \vdots \\ p_k\lambda_k + q_{k+2}\lambda_{k+2} \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} a_0\lambda_0 \\ a_1\lambda_0 \\ a_2\lambda_0 \\ \vdots \\ a_k\lambda_0 \end{bmatrix}$$

Thus we obtain the following expressions for the λ terms, expressed in fractions of p, q and a:

$\lambda_0 = 1\,\lambda_0 = a_0\lambda_0$ $\therefore\ a_0 = 1$

$\lambda_1 = \lambda_0/q_1 = a_1\lambda_0$ $\therefore\ a_1 = 1/q_1$

$\lambda_2 = (\lambda_1 - \lambda_0)/q_2 = a_2\lambda_0$ $\therefore\ a_2 = (a_1 - 1)/q_2$

$\vdots$

These can be generalized to:

$\lambda_k = (\lambda_{k-1} - p_{k-2}\lambda_{k-2})\,/q_k = a_k\lambda_0$ $\therefore\ a_k = (a_{k-1} - a_{k-2}p_{k-2})\,/\,q_k$

Note on the Size and Accuracy of the Finite Matrix

At this point, we can note that these formulas make it possible for the algorithm to produce finite matrix results which are in close approximation to infinite matrix results. To illustrate this, we can calculate the first few terms of a_i:

If $p_1 = 4/5$, then $q_1 = 1/5$. From these terms we can derive the following:

$p_2 = 4/6 \rightarrow q_2 = 2/6$ $\therefore\ a_1 = 1/q_1 = 5$

$p_3 = 4/7 \rightarrow q_3 = 3/7$ $\therefore\ a_2 = (a_1 - 1)/q_2 = 4\,/\,(2/6) = 12$

$p_4 = 4/8 \rightarrow q_3 = 4/8$ $\therefore\ a_3 = (a_2 - a_1p_1)/q_3 = (12 - (4/5)(6))\,/\,(3/7) = 56/3$

(These fractions can be reduced, but is it necessary to keep them in fractional form, for subsequent calculations.)

a_i will increase, peak, and then decrease. In this case, where $p_1 = 4/5$, we see that a_i peaks when i=4 and decreases thereafter. For example, when i=12:

$a_i = 0.105$

When a_i is less than a_0 ($a_i < a_0 = 1$) we can safely disregard the probabilities of higher terms as negligible. Stated formally, the probabilities for these terms become:

N := i+1, so:
$p_{i+1} = p_N = 0$
$q_{i+1} = q_N = 1$

In our first example, where $p_1 = 4/5$, we can set N = 12 for this reason. N chosen in this manner will give a close estimate of the actual value of an infinite matrix. (Although, as we will see in the next section, higher values of p_1 and N will give estimates that are much closer.)

Continuing the algorithm now:

We have a solution to $Q\Lambda = \Lambda$ in the form

$$\Lambda = \lambda_0 \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 1 \\ b_1 / c_1 \\ b_2 / c_2 \\ \vdots \\ b_N / c_N \end{bmatrix}$$

IV. We can find the least common multiple of the denominators, c_i, where i = 0, … , N

$\ell :=$ l.c.m. $\{ c_i \}$

Then we multiply Λ by ℓ, in order to remove the fractions and get whole numbers, which will be easier to work with. So:

$\ell\,\Lambda$ will consist of whole numbers.

$M := \ell \, (\sum_{i=0}^{N} a_i)$ will be the sum of these whole numbers.

Notice that $(\ell/M)(\sum_{i=0}^{N} a_i) = 1$

Which implies that $(\ell/M)\,\Lambda$ is a position vector.

And so $(\ell/M)\,\Lambda$ is the limiting position vector referred to in definition of the Euler-summable limiting matrix.

(Here the vector/scalar notation is abused slightly. We take ℓ/M as a scalar because $(\ell/M)\,\Lambda$ is a scalar multiple of the vector Λ. In our solution both Λ and the normalized vector $(\ell/M)\,\Lambda$ will be referred to as Λ. Notice that both are a solution to $Q\Lambda = \Lambda$.)

$$\Lambda = \lambda_0 \begin{bmatrix} \ell \\ \ell\, b_i / c_i \\ \vdots \\ \ell\, b_N / c_N \end{bmatrix} = \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_k \end{bmatrix} \qquad \text{where } n_i \in \text{integers.}$$

V. Compute $9 \times [N+1]$ table values; the "magnified values." Start on the left side, and compute a row from left to right, rounding to the nearest whole number. Many of the entries in the table will be whole numbers to begin with: Any rounding errors introduced in this step will be diminished by the factor M in step (VI).

Table 1 – Magnified Values

	$P(U^0_{i+1})$	$P(U^0_{i+1})p_i$	$P(U^0_{i+1})q_i p_{i-1}$	$P(U^0_{i+1})q_i q_{i-1} p_{i-2}$		$P(U^0_{i+1})q_i \ldots q_{i-5}$
n_0	$n_0 p_0$	$(n_0 p_0)p_1$	$(n_0 p_0)q_1 p_0$	0	…	0
…	…	…	…	…	…	…
n_{N-1}	$n_{N-1}p_{N-1}$	0	$(n_{N-1}p_{N-1})q_N p_{N-1}$	$(n_{N-1}p_{N-1})q_N q_{N-1} p_{N-2}$	…	$(n_{N-1}p_{N-1})q_N \ldots q_{N-5}$
n_N	0	0	0	0	…	0

VI. Divide the results of step (V) by M to normalize.

In theory, by normalizing at the very end of the calculations, the answers can be true to within (N/2)/M; accurate to several decimal places.

However, we will be choosing numeric parameters for this problem which are amenable to rapid manual calculation, rather than the highest accuracy. Also, we do not have a known, exact answer by which to gauge the accuracy of our results. The formula derived in Chapter 13 of the public essay:

(12) $P_n = 0.25n \times (1/2)^{n-1}$

is not rigorously proved to be the absolute probability formula for an infinite matrix solution.

This all suggests that the parameters chosen may result in a less accurate result than theory would indicate. We will see, however, that parameters can be found which produce absolute probabilities which consistently match those of (12) to within a 10% difference. (These probabilities also match those of "Monte Carlo" simulation, as discussed in the public essay in Chapter 16.) This is close enough to confirm the validity of those other, less rigorous, results.

VII. *Obtain the Solution of Problem (P1) -- Absolute Probabilities:* These nine normalized totals are precisely the following nine absolute event probabilities:

t_0/M = the absolute probability of *any* or *no* event (:= 1)
t_1/M = the absolute probability of experiencing any passage
t_2/M / (t_1/M) = the absolute probability of experiencing an *ex nihilo* passage
1 × t_3 / (t_1/M) = the absolute probability of experiencing a unitary 1-to-1 passage
2 × t_4 / (t_1/M) = the absolute probability of experiencing a 2-to-1 passage
3 × t_5 / (t_1/M) = the absolute probability of experiencing a 3-to-1 passage
4 × t_6 / (t_1/M) = the absolute probability of experiencing a 4-to-1 passage
5 × t_7 / (t_1/M) = the absolute probability of experiencing a 5-to-1 passage
6 × t_8 / (t_1/M) = the absolute probability of experiencing a 6^+-to-1 passage

VIII. *Obtain the Solution of Problem (P2) -- Relative Probability:* The probability of experiencing a merged passage, relative to that of experiencing a unitary passage, is just the sum of all absolute merger probabilities divided by the absolute unitary probability. Taking the formulae for these probabilities from step (VII), we get:

$(2t_4 + 3t_5 + 4t_6 + 5t_7 + 6t_8) / t_3 =$ the relative probability of merged to unitary passage

Parameters for a Preliminary Result

Please note: The following steps will produce a *preliminary* result. We will use parameters that give us an illustrative, and relatively concise, example of the algorithm "in action." Following on this example, a more *accurate* result will be presented in "Parameters for an Accurate Result."

So, proceeding on to a preliminary result. We will choose values of p_1 and N which will make for a relatively quick calculation. We will set p_1 = 4/5 and N = 12.

I. $p_1 = 4/5$

II.

$$Q = \begin{bmatrix} 0 & 1/5 & & & & & & & & & & & \\ 1 & 0 & 2/6 & & & & & & & & & & \\ & 4/5 & 0 & 3/7 & & & & & & & & & \\ & & 4/6 & 0 & 4/8 & & & & & & & & \\ & & & 4/7 & 0 & 5/9 & & & & & & & \\ & & & & 4/8 & 0 & 6/10 & & & & & & \\ & & & & & 4/9 & 0 & 7/11 & & & & & \\ & & & & & & 4/10 & 0 & 8/12 & & & & \\ & & & & & & & 4/11 & 0 & 9/13 & & & \\ & & & & & & & & 4/12 & 0 & & & \\ & & & & & & & & & 4/13 & \ddots & & \\ & & & & & & & & & & & \ddots & \end{bmatrix}$$

III. We run the equations for calculation of Λ in terms of a_i , where p_1 = 4/5 and N = 12:

$$\Lambda = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 12 \\ 56/3 \\ 64/3 \\ 96/5 \\ 128/9 \\ 2816/315 \\ 512/105 \\ 6656/35 \cdot 81 \\ 2^{11}/(5^2 \cdot 3^4) \\ 2^{12}/(11 \cdot 7 \cdot 5 \cdot 3^3) \\ 2^{14}/(11 \cdot 7 \cdot 5 \cdot 3^4) \end{bmatrix} = \begin{bmatrix} 1 \\ b_1/c_1 \\ b_2/c_2 \\ b_3/c_3 \\ b_4/c_4 \\ b_5/c_5 \\ b_6/c_6 \\ b_7/c_7 \\ b_8/c_8 \\ b_9/c_9 \\ b_{10}/c_{10} \\ b_{11}/c_{11} \\ b_{12}/c_{12} \end{bmatrix}$$

IV. ℓ := l.c.m. of $\{ c_i \}$

= l.c.m. of $\{ 1, 1, 3, 3, 5, 9, 315, 105, 35{\cdot}81, 5^2{\cdot}3^4, 11, 7{\cdot}5{\cdot}3^3, 11{\cdot}7{\cdot}5^2{\cdot}3^4 \}$

= $11{\cdot}7{\cdot}5^2{\cdot}3^4$

= 155,925

$$\ell\Lambda = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ n_8 \\ n_9 \\ n_{10} \\ n_{11} \\ n_{12} \end{bmatrix} = \begin{bmatrix} 155,925 \\ 779,625 \\ 1,871,100 \\ 2,910,600 \\ 3,326,400 \\ 2,993,760 \\ 2,217,600 \\ 1,393,920 \\ 760,320 \\ 366,080 \\ 157,696 \\ 61,440 \\ 16,384 \end{bmatrix} \qquad \mathrm{M} = \sum_{i=0}^{12} n_i = 17,010,850 \qquad \frac{\ell\Lambda}{\mathrm{M}} \approx \begin{bmatrix} 0.09 \\ 0.046 \\ 0.110 \\ 0.171 \\ 0.196 \\ 0.176 \\ 0.130 \\ 0.082 \\ 0.045 \\ 0.022 \\ 0.009 \\ 0.004 \\ 0.001 \end{bmatrix}$$

Note that because $\frac{\ell\Lambda}{\mathrm{M}}$ is normalized, its terms sum to 1.

V. Compute 9 × [N+1] table values; the “magnified values.” This is the table for N=12:

Table 2 – Explicit Formulae for Magnified Values

	$P(\mathrm{U}^0_{i+1})$	$P(\mathrm{U}^0_{i+1})p_i$	$P(\mathrm{U}^0_{i+1})q_i p_{i-1}$	$P(\mathrm{U}^0_{i+1})q_i q_{i-1} p_{i-2}$	$P(\mathrm{U}^0_{i+1})q_i q_{i-1} q_{i-2} p_{i-3}$	$P(\mathrm{U}^0_{i+1})q_i \ldots q_{i-3} p_{i-4}$	$P(\mathrm{U}^0_{i+1})q_i \ldots q_{i-4} p_{i-5}$	$P(\mathrm{U}^0_{i+1})q_i \ldots q_{i-5}$
n_0	$n_0 p_0$	$(n_0 p_0)p_1$	$(n_0 p_0)q_1 p_0$	0	0	0	0	0
n_1	$n_1 p_1$	$(n_1 p_1)p_2$	$(n_1 p_1)\, q_2 p_1$	$(n_1 p_1)\, q_2 q_1 p_0$	0	0	0	0
n_2	$n_2 p_2$	$(n_2 p_2)p_3$	$(n_2 p_2)\, q_3 p_2$	$(n_2 p_2)\, q_3 q_2 p_1$	$(n_2 p_2)\, q_3 q_2 q_1 p_0$	0	0	0
n_3	$n_3 p_3$	$(n_3 p_3)p_4$	$(n_3 p_3)\, q_4 p_3$	$(n_3 p_3)\, q_4 q_3 p_2$	$(n_3 p_3)\, q_4 q_3 q_2 p_1$	$(n_3 p_3)\, q_4 \ldots q_1 p_0$	0	0
n_4	$n_4 p_4$	$(n_4 p_4)p_5$	$(n_4 p_4)\, q_5 p_4$	$(n_4 p_4)\, q_5 q_4 p_3$	$(n_4 p_4)\, q_5 q_4 q_3 p_2$	$(n_4 p_4)\, q_5 \ldots q_2 p_1$	$(n_4 p_4)\, q_5 \ldots q_1 p_0$	0
n_5	$n_5 p_5$	$(n_5 p_5)p_6$	$(n_5 p_5)\, q_6 p_5$	$(n_5 p_5)\, q_6 q_5 p_4$	$(n_5 p_5)\, q_6 q_5 q_4 p_3$	$(n_5 p_5)\, q_6 \ldots q_3 p_2$	$(n_5 p_5)\, q_6 \ldots q_2 p_1$	$(n_5 p_5)\, q_6 \ldots q_1$
n_6	$n_6 p_6$	$(n_6 p_6)p_7$	$(n_6 p_6)\, q_7 p_6$	$(n_6 p_6)\, q_7 q_6 p_5$	$(n_6 p_6)\, q_7 q_6 q_5 p_4$	$(n_6 p_6)\, q_7 \ldots q_4 p_3$	$(n_6 p_6)\, q_7 \ldots q_3 p_2$	$(n_6 p_6)\, q_7 \ldots q_2$
n_7	$n_7 p_7$	$(n_7 p_7)p_8$	$(n_7 p_7)\, q_8 p_7$	$(n_7 p_7)\, q_8 q_7 p_6$	$(n_7 p_7)\, q_8 q_7 q_6 p_5$	$(n_7 p_7)\, q_8 \ldots q_5 p_4$	$(n_7 p_7)\, q_8 \ldots q_4 p_3$	$(n_7 p_7)\, q_8 \ldots q_3$
n_8	$n_8 p_8$	$(n_8 p_8)p_9$	$(n_8 p_8)\, q_9 p_8$	$(n_8 p_8)\, q_9 q_8 p_7$	$(n_8 p_8)\, q_9 q_8 q_7 p_6$	$(n_8 p_8)\, q_9 \ldots q_6 p_5$	$(n_8 p_8)\, q_9 \ldots q_5 p_4$	$(n_8 p_8)\, q_9 \ldots q_4$
n_9	$n_9 p_9$	$(n_9 p_9)p_{10}$	$(n_9 p_9)\, q_{10} p_9$	$(n_9 p_9)\, q_{10} q_9 p_8$	$(n_9 p_9)\, q_{10} q_9 q_8 p_7$	$(n_9 p_9)\, q_{10} \ldots q_7 p_6$	$(n_9 p_9)\, q_{10} \ldots q_6 p_5$	$(n_9 p_9)\, q_{10} \ldots q_5$
n_{10}	$n_{10} p_{10}$	$(n_{10} p_{10})p_{11}$	$(n_{10} p_{10})\, q_{11} p_{10}$	$(n_{10} p_{10})\, q_{11} q_{10} p_9$	$(n_{10} p_{10})\, q_{11} q_{10} q_9 p_8$	$(n_{10} p_{10})\, q_{11} \ldots q_8 p_7$	$(n_{10} p_{10})\, q_{11} \ldots q_7 p_6$	$(n_{10} p_{10})\, q_{11} \ldots q_6$
n_{11}	$n_{11} p_{11}$	0	$(n_{11} p_{11})\, q_{12} p_{11}$	$(n_{11} p_{11})\, q_{12} q_{11} p_{10}$	$(n_{11} p_{11})\, q_{12} q_{11} q_{10} p_9$	$(n_{11} p_{11})\, q_{12} \ldots q_9 p_8$	$(n_{11} p_{11})\, q_{12} \ldots q_8 p_7$	$(n_{11} p_{11})\, q_{12} \ldots q_7$
n_{12}	0	0	0	0	0	0	0	0

And here are the computed values:

Table 3 – Calculated Magnified Values

col. 0	col. 1	col. 2	col. 3	col. 4	col. 5	col. 6	col. 7	col. 8
155,925	155,925	124,740	31,185	0	0	0	0	0
779,625	623,700	415,800	166,320	41,850	0	0	0	0
1,871,100	1,247,400	712,800	356,400	142,560	35,640	0	0	0
2,910,600	1,663,200	831,600	475,200	237,600	95,040	23,760	0	0
3,326,400	1,663,200	739,200	462,000	264,000	132,000	52,800	13,200	0
2,993,760	1,330,560	532,224	354,816	221,760	126,720	63,360	25,344	6,336
2,217,600	887,040	322,560	225,792	150,528	94,080	53,760	26,880	13,440
1,393,920	506,880	168,960	122,880	86,016	57,344	35,840	20,480	15,360
760,320	253,440	77,982	58,486	42,535	29,775	19,850	12,406	12,406
366,080	112,640	32,183	24,756	18,567	13,503	9,452	6,304	7,877
157,696	45,056	12,015	9,440	7,262	5,446	3,961	2,773	4,159
61,440	16,384	0	4,369	3,433	2,641	1,980	1,440	2,521
16,384	0	0	0	0	0	0	0	0

VI. Divide the results of step (V.) by M to normalize the results. (We designate the sum total of column n as t_n.)

For each n-tuple passage multiply by the number of participants, n, and divide by the absolute probability of any passage, t_1.

For the special case of the 6^+-tuple passage, note that the total t_8 is computed by the corollary to Theorem (3).

$t_0/M = 17{,}010{,}850 / 17{,}010{,}850 = 1$

$t_1/M = 8{,}505{,}425 / 17{,}010{,}850 = 0.5$

$t_2/M / (t_1/M) = (3{,}970{,}064 / 17{,}010{,}850) / 0.5 = 0.4667684$

$1 \times t_3 / (t_1/M) = 1 \times (2{,}291{,}644 / 17{,}010{,}850) / 0.5 = 0.2694332$

$2 \times t_4 / (t_1/M) = 2 \times (1{,}215{,}841 / 17{,}010{,}850) / 0.5 = 0.2858978$

$3 \times t_5 / (t_1/M) = 3 \times (592{,}189 / 17{,}010{,}850) / 0.5 = 0.2088746$

$4 \times t_6 / (t_1/M) = 4 \times (264{,}763 / 17{,}010{,}850) / 0.5 = 0.1245149$

$5 \times t_7 / (t_1/M) = 5 \times (108{,}825 / 17{,}010{,}850) / 0.5 = 0.06397388$

$6 \times t_8 / (t_1/M) = 6 \times (62{,}099 / 17{,}010{,}850) / 0.5 = 0.04380663$

VII. Solution of Problem (P1) -- Absolute Probabilities:

Note again that the numeric parameters for this *preliminary* result were selected to be amenable to rapid manual calculation, rather than the highest accuracy. The following section, "Algorithm Steps for an Accurate Result," will produce the more accurate result.

Table 4 – Absolute Probabilities

passage type	absolute probability ≈
0-to-1 (*ex nihilo*)	0.467
1-to-1 (unitary)	0.269
2-to-1	0.286
3-to-1	0.209
4-to-1	0.125
5-to-1	0.064
6^{+}-to-1	0.044

VIII. Solution of Problem (P2) – Relative Probability:

$(2t_4 + 3t_5 + 4t_6 + 5t_7 + 6t_8) / t_3 =$ the relative probability of merged to unitary passage

$= (2\times1{,}215{,}841 + 3\times592{,}189 + 4\times264{,}763 + 5\times108{,}825 + 6\times62{,}099) / 2{,}291{,}644$

$= 2.70$

And so step (VIII) tells us that merged passages are 2.70 times as likely as unitary passages.

Printing the results of steps (VII) and (VIII) together, the solution tables for problems (P1) and (P2) are:

Table 5 – Absolute Probabilities

passage type	absolute probability ≈
0-to-1 (*ex nihilo*)	0.467
1-to-1 (unitary)	0.269
2-to-1	0.286
3-to-1	0.209
4-to-1	0.125
5-to-1	0.064

Table 6 – Relative Probability

ratio	relative probability ≈
Merged : unitary	2.70

Again, the numeric parameters for this preliminary result were selected to be amenable to rapid manual calculation, rather than the highest accuracy. The following section, “Parameters for an Accurate Result,” will produce the more accurate result, which we will review afterwards in “Discussion of Results”.

Parameters for an Accurate Result

Please note: The parameters in this section are chosen to give a result which is more accurate than the result calculated in the previous section, "Parameters for a Preliminary Result." All algorithm steps will be the same as before; so only the more accurate results themselves will be presented, and in an abbreviated manner, owing to the greater length of calculations required. Please refer to previous sections for explanations of each step.

Here we will choose values of p_1 and N which are higher than those used to obtain the preliminary result. We will set $p_1 = 9/10$ and N = 22.

I. $p_1 = 9/10$

II.

$$Q = \begin{bmatrix} 0 & 1/10 & & & & & & \\ 1 & 0 & 2/11 & & & & & \\ & 9/10 & 0 & 3/12 & & & & \\ & & 9/11 & 0 & 4/13 & & & \\ & & & 9/12 & 0 & 5/14 & & \\ & & & & 9/13 & 0 & & \\ & & & & & 9/14 & \ddots & \\ & & & & & & & \ddots \end{bmatrix}$$

III. Calculating a_i , where $p_1 = 9/10$ and N = 22:

$$\Lambda = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{15} \\ \lambda_{16} \\ \lambda_{17} \\ \lambda_{18} \\ \lambda_{19} \\ \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 99/2 \\ 162 \\ 3{,}159/8 \\ 15{,}309/20 \\ 3^9/16 \\ 3^{10}/35 \\ 3^{12}\cdot 17/35\cdot 2^7 \\ 3^{14}/35\cdot 2^6 \\ 3^{14}\cdot 19/7\cdot 5^2\cdot 2^8 \\ 3^{16}/11\cdot 7\cdot 5\cdot 2^6 \\ 3^{18}/11\cdot 5^2\cdot 2^{10} \\ 3^{19}/13\cdot 7\cdot 5^2\cdot 2^9 \\ 3^{21}\cdot 23/13\cdot 11\cdot 7^2\cdot 5^2\cdot 2^{11} \\ 3^{23}/13\cdot 11\cdot 7^2\cdot 5^3\cdot 2^8 \\ 3^{24}/13\cdot 11\cdot 7^2\cdot 5\cdot 2^{15} \\ 3^{26}/17\cdot 11\cdot 7^2\cdot 5^3\cdot 2^{14} \\ 3^{29}/17\cdot 13\cdot 11\cdot 7^2\cdot 5^3\cdot 2^{16} \\ 3^{28}/19\cdot 17\cdot 13\cdot 11\cdot 7\cdot 5^3\cdot 2^{14} \\ 3^{30}\cdot 29/19\cdot 17\cdot 13\cdot 11\cdot 7^2\cdot 5^4\cdot 2^{18} \\ 3^{32}/19\cdot 17\cdot 13\cdot 11\cdot 7^3\cdot 5^3\cdot 2^{17} \\ 3^{33}/19\cdot 17\cdot 13\cdot 11\cdot 7^3\cdot 5^4\cdot 2^{18} \end{bmatrix} = \begin{bmatrix} 1 \\ b_1/c_1 \\ b_2/c_2 \\ b_3/c_3 \\ b_4/c_4 \\ b_5/c_5 \\ b_6/c_6 \\ b_7/c_7 \\ b_8/c_8 \\ b_9/c_9 \\ b_{10}/c_{10} \\ b_{11}/c_{11} \\ b_{12}/c_{12} \\ b_{13}/c_{13} \\ b_{14}/c_{14} \\ b_{15}/c_{15} \\ b_{16}/c_{16} \\ b_{17}/c_{17} \\ b_{18}/c_{18} \\ b_{19}/c_{19} \\ b_{20}/c_{20} \\ b_{21}/c_{21} \\ b_{22}/c_{22} \end{bmatrix}$$

IV. $\ell = 11\cdot 7\cdot 5^2\cdot 2^8 = 492{,}800$

$$\ell\Lambda = \begin{bmatrix} 492{,}800 \\ 4{,}928{,}000 \\ 24{,}393{,}600 \\ 79{,}833{,}600 \\ 194{,}594{,}400 \\ 377{,}213{,}760 \\ 606{,}236{,}400 \\ 831{,}409{,}920 \\ 993{,}794{,}670 \\ 1{,}052{,}253{,}180 \\ 999{,}640{,}521 \\ 860{,}934{,}420 \\ 677{,}985{,}856 \\ 491{,}726{,}005 \\ 330{,}478{,}192 \\ 206{,}908{,}085 \\ 121{,}235{,}206 \\ 66{,}750{,}678 \\ 34{,}659{,}006 \\ 17{,}025{,}477 \\ 7.935{,}088 \\ 3{,}518{,}019 \\ 1{,}055{,}406 \end{bmatrix} \qquad M = \sum_{i=0}^{22} n_i = 7{,}985{,}002{,}289 \qquad \frac{\ell\Lambda}{M} \approx \begin{bmatrix} 0.000 \\ 0.001 \\ 0.003 \\ 0.010 \\ 0.024 \\ 0.047 \\ 0.076 \\ 0.104 \\ 0.124 \\ 0.132 \\ 0.125 \\ 0.108 \\ 0.085 \\ 0.062 \\ 0.041 \\ 0{,}026 \\ 0.015 \\ 0.008 \\ 0.004 \\ 0.002 \\ 0.001 \\ 0.000 \\ 0.000 \end{bmatrix}$$

V. Compute 9 × [N+1] table values; the “magnified values.” This is the table for N=22:

Table 7 – Calculated Magnified Values

col. 0	col. 1	col. 2	col. 3	col. 4	col. 5	col. 6	col. 7	col. 8
492,800	492,800	443,520	49,280	0	0	0	0	0
4,928,000	4,435,200	3,628,800	725,760	80,640	0	0	0	0
24,393,600	19,958,400	14,968,800	4,082,400	816,480	90,720	0	0	0
79,833,600	59,875,200	41,452,062	13,817,354	3,768,369	753,674	83,742	0	0
194,594,400	134,719,200	86,605,200	33,309,692	11,103,231	3,028,154	605,631	67,292	0
377,213,760	242,494,560	145,496,736	62,355,744	23,982,978	7,994,326	2,180,271	436,054	48,450
606,236,400	363,741,840	204,604,785	95,482,233	40,920,957	15,738,830	5,246,277	1,430,803	317,956
831,409,920	467,668,808	247,588,983	123,794,492	57,770,763	24,758,898	9,522,653	3,174,218	1,058,073
993,794,670	526,126,590	263,063,295	139,268,803	69,634,402	32,496,054	13,926,880	5,356,492	2,380,663
1,052,253,180	526,126,590	249,217,858	138,454,366	73,299,370	36,649,685	17,103,186	7,329,937	4,072,187
999,640,521	473,513,931	213,081,269	123,362,840	68,534,911	36,283,188	18,141,594	8,466,077	5,644,051
860,934,420	387,420,489	166,037,352	99,622,411	57,676,133	32,042,296	16,963,569	8,481,784	6,596,943
677,985,856	290,565,361	118,867,650	73,584,736	44,150,841	25,561,013	14,200,563	7,517,945	6,682,618
491,726,005	201,160,638	78,715,032	50,091,384	31,008,952	18,605,371	10,771,531	5,984,184	5,984,184
330,478,192	129,317,553	48,494,083	31,626,576	20,126,003	12,458,954	7,475,372	4,327,847	4,808,719
206,908,085	77,590,532	27,932,591	18,621,728	12,144,605	7,728,385	4,784,238	2,870,543	3,508,441
121,235,206	43,644,674	15,107,772	10,273,285	6,848,857	4,466,646	2,842,411	1,759,588	2,346,117
66,750,678	23,106,004	7,702,001	5,332,155	3,625,865	2,417,243	1,576,463	1,003,204	1,449,072
34,659,006	11,553,002	3,713,465	2,613,179	1,809,124	1,230,204	820,136	534,871	832,022
17,025,477	5,472,475	1,698,354	1,213,110	853,670	591,002	401,882	267,921	446,535
7,935,088	2,462,614	738,784	534,982	382,130	268,906	186,166	126,593	225,054
3,518,019	1,055,406	0	316,622	229,278	163,770	115,245	79,785	150,706
1,055,406	0	0	0	0	0	0	0	0

VI. Normalize the results:

$t_0/\mathrm{M} = 7{,}985{,}002{,}289 / 7{,}985{,}002{,}289 = 1$

$t_1/\mathrm{M} = 3{,}992{,}501{,}145 / 7{,}985{,}002{,}289 = 0.5$

$t_2/\mathrm{M} / (t_1/\mathrm{M}) = (1{,}939{,}158{,}392 / 7{,}985{,}002{,}289) / 0.5 = 0.485700$

$1 \times t_3 / (t_1/\mathrm{M}) = 1 \times (1{,}028{,}533{,}172 / 7{,}985{,}002{,}289) / 0.5 = 0.257616$

$2 \times t_4 / (t_1/\mathrm{M}) = 2 \times (528{,}767{,}559 / 7{,}985{,}002{,}289) / 0.5 = 0.26488$

$3 \times t_5 / (t_1/\mathrm{M}) = 3 \times (263{,}327{,}319 / 7{,}985{,}002{,}289) / 0.5 = 0.197865$

$4 \times t_6 / (t_1/\mathrm{M}) = 4 \times (126{,}947{,}810 / 7{,}985{,}002{,}289) / 0.5 = 0.127188$

$5 \times t_7 / (t_1/\mathrm{M}) = 5 \times (59{,}215{,}138 / 7{,}985{,}002{,}289) / 0.5 = 0.074155$

$6 \times t_8 / (t_1/\mathrm{M}) = 6 \times (46{,}551{,}791 / 7{,}985{,}002{,}289) / 0.5 = 0.07002$

VII. Solution of Problem (P1) -- Absolute Probabilities:

Note again that this result is more accurate than the preliminary result calculated previously. This more accurate result, and the result in (VIII) below, will be reviewed hereafter in "Discussion of Results".

Table 8 – Absolute Probabilities

passage type	absolute probability
ex nihilo	0.486
unitary	0.258
2-to-1	0.265
3-to-1	0.198
4-to-1	0.127
5-to-1	0.074
6^+-to-1	0.070

VIII. Solution of Problem (P2) -- Relative Probability:

$(2t_4 + 3t_5 + 4t_6 + 5t_7 + 6t_8) / t_3 =$ the relative probability of merged to unitary passage

$= (2\times528{,}767{,}559 + 3\times263{,}327{,}319 + 4\times126{,}947{,}810 + 5\times59{,}215{,}138 + 6\times46{,}551{,}791) / 1{,}028{,}533{,}172$

$= 2.85$

And so step (VIII) tells us that merged passages are 2.85 times as likely as unitary passages.

Printing the results of steps (VII) and (VIII) together, the solution tables for problems (P1) and (P2) are:

Table 9 – Absolute Probabilities

passage type	absolute probability
0-to-1 (*ex nihilo*)	0.486
1-to-1 (unitary)	0.258
2-to-1	0.265
3-to-1	0.198
4-to-1	0.127
5-to-1	0.074

Table 10 – Relative Probability

ratio	relative probability
Merged : unitary	2.85

These results are intended to be more accurate than the preliminary results obtained in the previous section. For a side-by-side comparison of several result sets, see "Discussion of Results", which follows.

Discussion of Results

In the previous two sections, two different choices of starting parameters p_1 and N were used, and results were obtained for both. Altogether, a total of five combinations of p_1 and N have been tried during preparation of this appendix document. Results for the five different parameter combinations are tabularized below:

Table 11 – Comparison of Results

*

most accurate

	p_1=3/4, N=11	p_1=4/5, N=12	p_1=4/5, N=18	p_1=9/10, N=12	p_1=9/10, N=22	*theoretical*
ex nihilo	0.455	0.467	0.467	0.458	0.486	0.500
unitary	0.278	0.269	0.269	0.283	0.258	0.250
2-to-1	0.299	0.286	0.286	0.281	0.265	0.250
3-to-1	0.213	0.209	0.209	0.200	0.198	0.188
4-to-1	0.120	0.125	0.125	0.120	0.127	0.125
5-to-1	0.055	0.064	0.064	0.064	0.074	0.078
merged/unitary	2.60	2.70	2.70	2.51	2.85	3.00

Table 11 brings together five sets of calculated results for comparison. Each row is labeled at left with the type of passage event probability calculated. Before comparing the numbers, we should review the meaning of each of these probabilities:

The only participant in an *ex nihilo* passage is the person born; no one "passes" to the recipient of an *ex nihilo* passage.

The other absolute probabilities listed (unitary, 2-to-1, 3-to-1, 4-to-1, 5-to-1) are the probabilities that a person will pass through each of those particular passage types to a newborn. The selected algorithm calculates individual absolute probabilities only out to 5-to-1 merged passages. As a result, the table displays only the n-to-1 passages out to n=5. The algorithm can be extended to higher-order n, but this is not necessary for the present purpose.

The bottom row displays a ratio: the probability that a person will experience a merged passage, divided by the probability of a unitary passage. This is the only ratio in the table, and we should recall that its formula makes use of the aggregate 6^+-to-1 absolute probability, which is not printed in this table.

Now, comparing the numbers to prediction:

At far right the "theoretical" probabilities are listed. These values are not proved to be the limiting and authoritative ones, but the informal probability argument provided in the essay at mbdefault.org in Chapters 13-16 suggests that they are. For now we will take them as the theoretical values.

The top row of each column displays the parameters used to generate each computed result set. The discussion works across columns from left to right, on the following page:

Column 1 (p_1=3/4, N=11)
This result has been computed from the lowest combination of parameter values: 3/4 is the smallest value of p_1 we have used, and 11 is the smallest value of N we have used. The resulting matrix Q was small, and its matrix element values were low fractions. This made for a quick, but inaccurate, calculation. If we compare the results listed in Column 1 against the theoretical values, we see that this result set is the farthest of the five from theory.

Column 2 (p_1=4/5, N=12)
This result is just the "preliminary result" which we presented in detail in "Parameters for a Preliminary Result." Here we've increased both p_1 and N in hopes of obtaining a better fit to theory. The larger matrix, with higher fractions, has indeed improved the fit to theory. All computed values are closer to their theoretical counterparts.

Column 3 (p_1=4/5, N=18)
Here we've kept p_1 unchanged at 4/5, and increased N to 18; in order to see what higher-order matrix elements alone might contribute to the solution. As it turns out, their contribution is negligible: all values in Column 3 are the same as those of Column 2. (Differences appear in the raw numbers only past the third significant digit.) This is because, as we stated in Section III of "Details of Algorithm Steps," when $a_i < 1$ the terms are negligible. In this case, $a_{18} = 0.000000$, and truly negligible. This suggests that if we are to increase the accuracy of the results further, it will necessary to increase p_1 instead.

Column 4 (p_1=9/10, N=12)
Here we have increased p_1 to 9/10. But because the matrix is small (N=12) the accuracy is not better than that of the previous three attempts. And so we can conclude that it will be necessary to increase N as well.

Column 5 (p_1=9/10, N=22)
Here we have increased p_1 and N to their maximum values of the trial. Note that these values are just those of the "more accurate result" outlined in "Parameters for an Accurate Result." This lengthiest calculation has indeed produced a more accurate result. The values in Column 5 are clearly the best fit to the theoretical values. And so Column 5 is marked with an asterisk (*****) as "most accurate."

End of Appendix A.